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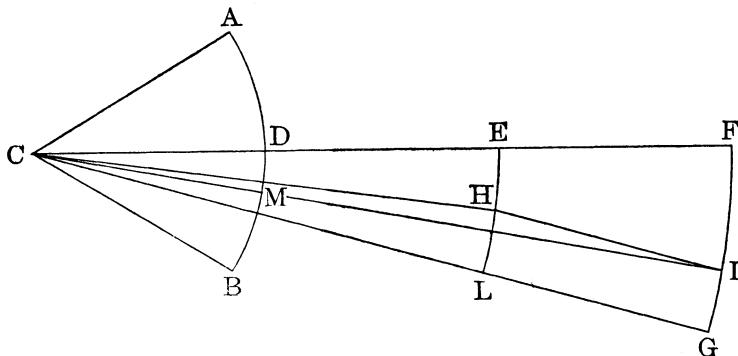
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## THE APPROXIMATE INSCRIPTION OF CERTAIN REGULAR POLYGONS.

By PROF. H. A. HOWE, Denver, Col.

1. To inscribe a regular nonagon in a circle.—Let  $ACB$  be a central angle of  $60^\circ$  in the given circle. Bisect  $ACB$  by  $CF$ , laying off  $DE$  and  $EF$  equal to  $CD$ .



Bisect  $FCB$  by  $CG$ , and draw the arcs  $EL$  and  $FG$  from  $C$  as a centre. Bisect  $FCG$ , thus determining  $H$ , the mid-point of  $EL$ . From  $H$  draw  $HI$  parallel to  $CG$ , and from the point  $I$  where it cuts  $FG$  draw  $IC$ . Then will  $ICG$  be nearly equal to one third of  $FCG$ , or  $5^\circ$ . By solving the triangle  $HIC$  we find that  $CIH = ICG = 4^\circ 59' 31''.37$ . Hence  $ACM = 40^\circ 0' 28''.63$ , and a chord drawn from  $A$  to  $M$  will be very nearly equal to the side of a regular nonagon inscribed in the circle whose radius is  $AC : 9 \times 28''.63 = 257''.67$ , which is  $\frac{1}{5030}$  of the entire circumference. If one half the angle  $FCI$  were subtracted from an angle of  $45^\circ$ , an angle of  $39^\circ 59' 45''.68$  would result, which is a closer approximation to  $40^\circ$ .

It is evident that there are many ways of getting angles more nearly equal to  $5^\circ$ ; we give a few, each of which depends upon the trisection of a small angle, together with successive bisections of angles; thus:

$$5^\circ = \frac{15}{2}^\circ - \frac{1}{3} \cdot \frac{15}{2}^\circ \quad (A)$$

$$= \frac{1}{4}^\circ + \frac{1}{3} \cdot \frac{1}{4}^\circ \quad (B)$$

$$= \frac{15}{4}^\circ + \frac{15}{8}^\circ - \frac{1}{3} \cdot \frac{15}{8}^\circ \quad (C)$$

$$= \frac{15}{4}^\circ + \frac{15}{16}^\circ + \frac{1}{3} \cdot \frac{15}{16}^\circ \quad (D)$$

$$= \frac{15}{4}^\circ + \frac{15}{16}^\circ + \frac{15}{32}^\circ - \frac{1}{3} \cdot \frac{15}{32}^\circ \quad (E)$$

$$= \frac{1}{4}^\circ + \frac{1}{16}^\circ + \frac{1}{64}^\circ + \frac{1}{3} \cdot \frac{1}{64}^\circ, \quad (F)$$

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If the trisections included in solutions  $(A), \dots (F)$  be performed in the same manner in which  $FCG$  was trisected (getting  $ICG$  as the approximate value of  $\frac{1}{3}FCG$ ), it is easy to compute the error of each solution in the following way:—

Let  $FCG$  now denote any one of these angles to be trisected; denote  $ICG$  by  $x$  and  $HCG$  by  $y$ ; let  $2F(y)$  denote  $y - \sin y$ , and  $2F(x)$  denote  $x - \sin x$ .

Since  $\sin y = \frac{3}{2} \sin x$ , we have

$$y - 2F(y) = \frac{3}{2}x - \frac{3}{2} \cdot 2F(x). \quad (1)$$

By development,

$$2F(y) = \frac{1}{6} \sin^3 y + \frac{3}{40} \sin^5 y + \frac{5}{112} \sin^7 y + \frac{35}{1152} \sin^9 y + \dots \quad (2)$$

Likewise, substituting  $\frac{3}{2} \sin y$  for  $\sin x$ , we get

$$2F(x) = \frac{4}{81} \sin^3 y + \frac{4}{405} \sin^5 y + \frac{40}{15309} \sin^7 y + \frac{140}{177147} \sin^9 y + \dots \quad (3)$$

Hence

$$\frac{2F(x)}{2F(y)} = \frac{8}{27} - (\frac{2}{27} \sin^2 y + \frac{1549}{51030} \sin^4 y + \frac{651827}{4133430} \sin^6 y + \dots) = \frac{8}{27} - \varphi(y), \quad (4)$$

where  $\varphi(y)$  denotes the series in the ( ).

Find the value of  $F(x)$  from (4), substitute it in (1), and reduce, obtaining

$$y - \frac{3}{2}x = \frac{5}{9} \cdot 2F(y) + \frac{3}{2} \varphi(y) \cdot 2F(y),$$

or

$$\frac{2}{3}y - x = \frac{10}{27} \cdot 2F(y) + \varphi(y) \cdot 2F(y). \quad (5)$$

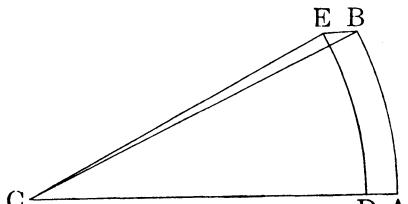
Now  $\frac{2}{3}y - x$  is the error of  $x$ , and (5) furnishes an easy method of computing this error. Thus we find the following table:—

Solution.	$2y$	Error of $x$ .	$\frac{9}{1,296,000} \times \text{error of } x$
(A)	7.5°	''	I : 40,290 ±
(B)	3.75	0.4462	I : 322,700 ±
(C)	1.875	0.0558	I : 2,583,000 ±
(D)	0.9375	0.00698	I : 20,660,000 ±
(E)	0.46875	0.00087	I : 165,000,000 ±
(F)	0.234375	0.00011	I : 1,300,000,000 ±

It thus becomes apparent that it is possible to find a third of any acute angle without much labor, and attain a result the theoretical error of which is far less than the unavoidable errors of the most careful draughtsmen with the best drawing instruments manufactured. When the angle is between  $45^\circ$  and

$90^\circ$ , it is best to find one third of its complement, and subtract it from  $30^\circ$ . If the given angle is obtuse it is best to find a third of its supplement, and subtract it from  $60^\circ$ .

2. *The Inscription of a Regular Polygon of Eleven Sides.—1st Method.*—



Divide the radius  $CA$  of the given circle into twelve equal parts, of which  $CD$  contains eleven. Construct  $ACB = 30^\circ$ . Draw  $AB$  and  $DE$ . From  $B$  draw  $BE$  parallel to  $AC$ . Draw  $CE$ . Then  $ECD = 33^\circ 3' 20''$ , which is one per cent. larger than  $360^\circ : 11$ . If the arc  $ED$  had been

drawn with a radius of  $11.1$ ,  $ECD$  would have been equal to  $32^\circ 43' 13''.60$ , which is  $\frac{1}{4793}$  smaller than  $360^\circ : 11$ .

2d Method.—If a triangle be formed of sides 6, 10, 11, the angle opposite 6 will be  $32^\circ 45' 50''$ , which is  $\frac{1}{897}$  larger than  $360^\circ : 11$ .

3. *The Inscription of a Regular Polygon of Thirteen Sides.*—In the preceding figure let  $CD = 12$ , and  $CA = 13$ , and the angle  $ECD = 30^\circ$ ,  $EB$  being parallel to  $AC$ . Then  $ACB = 27^\circ 29' 11''.14$ , which is  $\frac{1}{1345}$  smaller than  $360^\circ : 13$ . If  $CA$  be taken as  $12\frac{1}{11}$ ,  $ACB$  will be  $15''.27$  larger than desired, or  $\frac{1}{6529}$  too large.

4. *The Multisection of Small Angles.*—If we wish to find one fifth of  $ACB$ , by successive bisections we obtain  $ACG = \frac{1}{4}BCA$ . Lay off  $CH = 4$  and  $CA = 5$ , and draw the arcs  $BA$  and  $HK$ . From  $E$ , where  $CG$  cuts  $HK$ , draw  $EF$  parallel to  $CA$ .  $FCA$  is nearly one fifth of  $BCA$ . Denote  $GCA$  by  $\alpha$ , and  $FCA$  by  $x$ . By reasoning similar to that employed before we may show that

$$\frac{4}{5}\alpha - x = \left( \frac{3.6}{125} + \frac{12.96}{15625} \sin^2 \alpha + \dots \right) 2F(\alpha).$$

If  $BCA = 15^\circ$ ,  $FCA$  is  $2^\circ 59' 57''.22$ , which is  $2''.78$  too small.

To find  $\frac{1}{7}$  of  $BCA$ . By successive bisections, find  $FCA = \frac{1}{8}BCA$ . Lay off  $CA = 8$  and  $CH = 7$ , and draw  $BA$  and  $KH$ ; also draw  $FE$  parallel to  $CA$ .  $ECA$  is nearly equal to  $\frac{1}{7}BCA$ . Denoting  $FCA$  by  $\alpha$ , and  $ECA$  by  $x$ , we find

$$\frac{8}{7}\alpha - x = \left( -\frac{12.0}{343} - \frac{34.56}{16807} \sin^2 \alpha - \dots \right) 2F(\alpha).$$

If  $BCA = 15^\circ$ ,  $ECA = 2^\circ 8' 34''.71$ , which is  $0''.42$  too large.

By combinations of the methods which have been set forth one may easily find  $\frac{1}{9}$ ,  $\frac{1}{10}$ ,  $\frac{1}{11}$ , etc. of any small angle with considerable accuracy. These methods are so simple that it is quite probable that they are not new; but I have never heard of them.

